Limits---Getting Down to Details
Written for use in Calculus I
by
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Motivation and Introduction: We recently began an intuitive discussion of the concept of a limit. This idea came up in our discussion as we tried to see how to compute the slope of a general, curved graph. (Or, equivalently, the instantaneous rate of change of a function at a particular point.) We begin by briefly re-capping that set of ideas here.

Consider the function $g(x)=\dfrac{x}{1+x^2}$ whose graph is shown in Figure 1. Suppose that we want to know the slope of the graph of g at $x = 2$. We cannot compute the slope of the tangent line to the graph directly, since we need two points to compute a slope, so we approximate the slope by computing the slope of the secant line obtained by joining (2, g (2)) with a nearby point on the graph. (See Figure 2.)

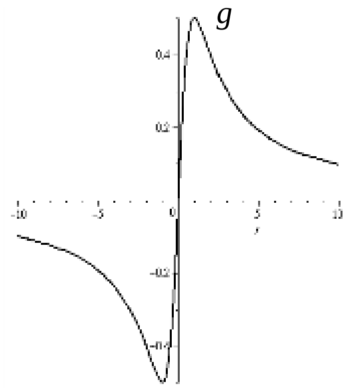

Figure 1

This approximation can be improved by picking another point on the graph even closer to (2, g(2)) and recomputing the slope. This new approximation is better, but is still only an approximation. So we pick another point that is even closer and recompute the slope again. We repeat this process over and over again, each time getting a better approximation to the actual slope of the graph, but each time being able to improve by choosing yet another point.[1] This sequence of numbers is getting closer and closer to the number we actually want to calculate, which is the slope of the tangent line to the graph of g at $x = 2$. Some values in which the "nearby point" approaches both from the right and from the left are found in Table 1.

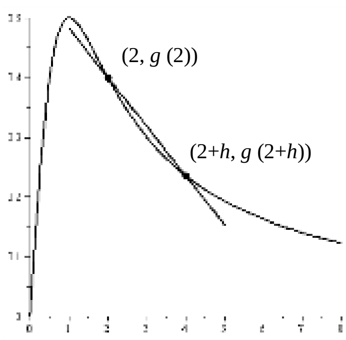

Figure 2

We notice a trend: our approximating slopes are getting "closer and closer" to -.12. We make the intuitive leap that the *actua*l slope of the graph at $x = 2$ is -.12. We denote the slope of the graph at $x = 2$ by $g'(2)$, and we say that $g'(2) = -.12$ is *the limit of the sequence of approximating slopes.* Symbolically, we represent this idea by the notation:

$$g'(2)=\lim_{h\to 0}\frac{g(2+h)-g(2)}{h}=-.12.$$

h	Secant Slope
1	-0.10000
0.1	-0.11830
0.01	-0.11984
0.001	-0.11998
-1	-0.10000
-0.1	-0.12148
-0.01	-0.12016
-0.001	-0.12002

Table 1: *h* is the distance between 2 and the "nearby point."

[1] Recall the alternate viewpoint: each of these approximating slopes represents the average rate of change of the function over the interval. When we take the average over shorter and shorter intervals including our point, each average gets closer and closer to the actual (instantaneous) rate of change of our function at the point x=2.

To this point, our approach to the concept of limit has been entirely intuitive, motivated by our desire to study rates of change and slopes. The purpose of this unit is to introduce some precision into our discussion of limits and to expand our point of view on the subject.

Graphical Discussion of the Difference Quotient Problem: Based on the numerical experiment above, we guessed that $g'(2) = -.12$. How can we be sure? Let us look more closely at the mathematical situation. We were considering values of the difference quotient:

$$d(h) = \frac{g(2+h) - g(2)}{h} = \frac{\dfrac{2+h}{1+(2+h)^2} - \dfrac{2}{5}}{h}$$

By doing some algebra "off-stage," we see that for $h \neq 0$, this expression is equal to

$$d(h) = -\frac{1}{5}\left(\frac{3+2h}{5+4h+h^2}\right).$$

Why do I specify that *h* cannot be zero? Because the original expression is not defined when *h* = 0, and the simplified expression is. In fact, we are interested in the behavior, around 0, of two closely related functions:

$$d(h) = \begin{cases} -\dfrac{1}{5}\left(\dfrac{3+2h}{5+4h+h^2}\right) & h \neq 0 \\ \text{undefined} & h = 0 \end{cases} \quad \text{and} \quad D(h) = -\frac{1}{5}\left(\frac{3+2h}{5+4h+h^2}\right) \text{ for all } h$$

The graphs of these two functions are shown in Figure 3.

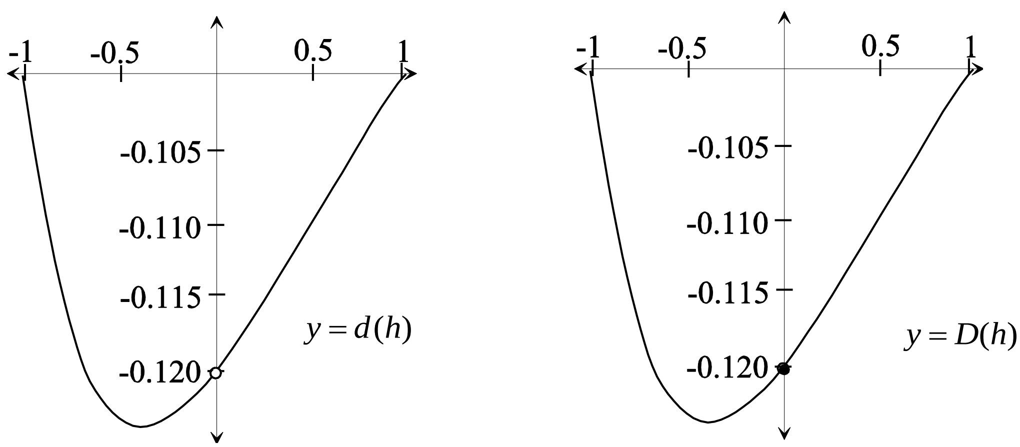

Figure 3: Notice that *d* is not defined at *h*=0, but *D* is. Otherwise they are the same.

If we ask about the "trend" in the outputs of these two functions as *h* gets closer and closer to 0 but never actually reaches 0, we will clearly observe the same trend in both functions. That is,

$$\lim_{h\to 0} d(h) = \lim_{h\to 0} D(h).$$

Though these two functions are not equal because d is not defined at h = 0 and D is, they *have the same limit* as h approaches 0. Since $D(h)$ is clearly approaching $D(0)=\dfrac{-3}{25}=-.12$ as h approaches 0, [2] we can conclude that, indeed,

$$g'(2)=\lim_{h\to 0} d(h)=-.12,$$

as we originally guessed by looking at the results of our numerical experiment. Only now we are certain of the result. <u>Our conclusion is no longer tentative</u>.

Defining the Limit

In the derivative problem, we said that $g'(2)=\lim_{h\to 0} d(h)=-.12$. This means that as the input h gets closer and closer to zero <u>without ever reaching 0</u>, the output $d(h)$ will get closer and closer to -.12. We can now generalize this idea.

Suppose we have a function f that is defined for all values near $x = a$, but not necessarily at $x = a$. Then we say that

> The limit as x approaches a of $f(x)$ is L, provided that as x gets closer and closer to a without ever reaching a, $f(x)$ gets closer and closer to L.

More informally we might say that

As the inputs of f approach a, the outputs of f approach L.

(Here we are leaving out the phrase "without ever reaching a," though it is absolutely crucial, as we will discuss shortly. Just remember that we often leave it out when we speak informally, but that it is always implicit in any discussion of limits.)

There are different ways of denoting limits symbolically. We have already seen a version of

$$\lim_{x\to a} f(x)=L.$$

We read this as "the limit as x goes to a of $f(x)$ is L."

Or we can write

$$f(x)\to L \text{ as } x\to a.$$

We read this as "$f(x)$ approaches L as x approaches a."

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[2] The fact that $\lim_{h\to 0} D(h)=D(0)$ seems intuitively obvious from the graph. This is true because the function D is *continuous* at h = 0. We will discuss continuity more carefully later on and put some "teeth" into our intuition.

Consider the graph of a function *f* shown in Figure 3. Observe that as the values of *x* approach 2 (from both the right and the left), the corresponding values of $f(x)$ approach 4.

$$\text{Thus } \lim_{x\to 2} f(x) = 4.$$

Notice also that $f(2) = 5$ but this does not affect the limit in any way, since our definition tells us to consider values $f(x)$ when *x* is close to 2, but explicitly forbids us from considering the value of $f(x)$ when *x* is equal to 2. Limits are about the behavior of the function around a point. When we consider the continuity of functions later on, we will compare and contrast the behavior of the function near the point with the behavior of the function at the point

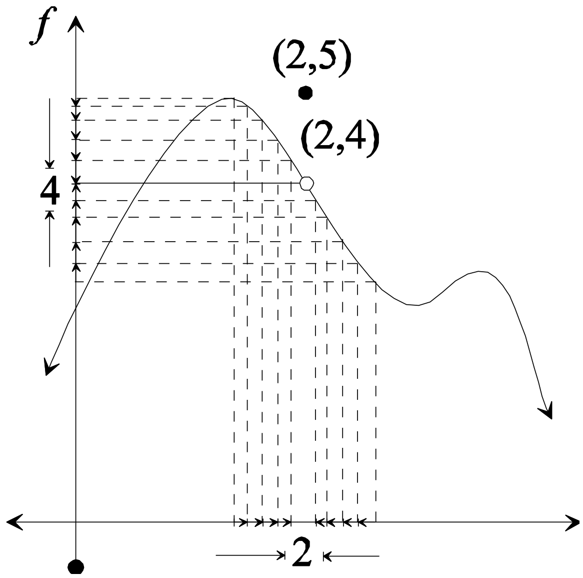
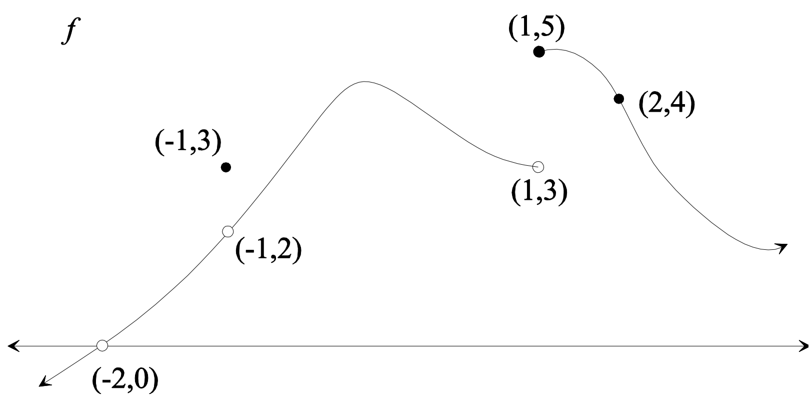


Figure 3

Exercise: Consider the function *f* whose graph is shown below.



Answer the following questions about the function *f*.

a. What is $f(-2)$? What is $\lim_{x\to -2} f(x)$?
b. What is $f(-1)$? What is $\lim_{x\to -1} f(x)$?
c. What is $f(2)$? What is $\lim_{x\to 2} f(x)$?
d. What is $f(1)$? What can you say about $\lim_{x\to 1} f(x)$? (In particular, is it true that as the input values approach 1, the output values approach some single value *L*?)